

## ON CERTAIN COMPUTATIONS OF PISOT NUMBERS

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**ABSTRACT.** This paper presents two algorithms on certain computations about Pisot numbers. Firstly, we develop an algorithm that finds a Pisot number  $\alpha$  such that  $\mathbb{Q}[\alpha] = \mathbb{F}$  given a real Galois extension  $\mathbb{F}$  of  $\mathbb{Q}$  by its integral basis. This algorithm is based on the lattice reduction, and it runs in time polynomial in the size of the integral basis. Next, we show that for a fixed Pisot number  $\alpha$ , one can compute  $[\alpha^n] \pmod{m}$  in time polynomial in  $(\log(mn))^{O(1)}$ , where  $m$  and  $n$  are positive integers.

## INTRODUCTION

A Pisot number (or Pisot-Vijayaraghavan number) is a real algebraic integer greater than 1, whose Galois conjugates over  $\mathbb{Q}$  are all of modulus strictly less than 1. Generally, given a real number  $\varepsilon > 0$ , an algebraic integer is called an  $\varepsilon$ -Pisot number if all its Galois conjugates have modulus less than  $\varepsilon$  [7]. The most famous Pisot number is the golden ratio  $\frac{1+\sqrt{5}}{2}$ . Pisot numbers have many interesting properties in their own right. Not surprisingly, they have many applications in diverse areas, such as harmonic analysis, statistics and the Diophantine approximation. For an introduction to the Pisot numbers, we refer the reader to the books [13] and [2].

In this paper, we study two computational problems about Pisot numbers: one is to find a Pisot number generating a real Galois number field, and the other is to compute the modular exponentiation of a Pisot number.

There are several known ways to find Pisot numbers in different situations. To name a few: Dufresnoy and Pisot [6] developed a method to find all Pisot numbers in the real interval  $[1, \frac{1+\sqrt{5}}{2} + \varepsilon]$ , where  $0 < \varepsilon < 0.0004$ . Boyd [4] modified Dufresnoy and Pisot's algorithm to determine all the Pisot numbers in an interval of the real line  $[\alpha, \beta]$  if there are finitely many in the interval. Bell and Hare [9] gave a classification of some Pisot-Cyclotomic numbers. Utilizing the Lenstra-Lenstra-Lovasz (LLL) algorithm [1], we show the following result.

**Theorem 0.0.1.** *Let  $\mathbb{F}$  be a real Galois extension over  $\mathbb{Q}$  given by its integral basis  $\beta_1, \dots, \beta_k$ . There exists a polynomial time algorithm to determine integers  $a_1, a_2, \dots, a_k$  such that*

$$\alpha = a_1\beta_1 + \dots + a_k\beta_k$$

*is a Pisot number and  $\mathbb{Q}[\alpha] = \mathbb{F}$ .*

**Remark 0.0.2.** *There are many ways to represent an algebraic number [5]. For example, one can represent an algebraic number by its minimal polynomial and a complex number, which is closer to the number than any of its conjugates. The*

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size of an algebraic number is defined to be the size of its minimal polynomial. The size of an integral polynomial  $\sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$  ( $a_d \neq 0$ ) is defined to be  $d \log(\max_i \{|a_i| + 1\})$ . In many of our examples, we work in the real sub-field of a cyclotomic field  $\mathbb{Q}(\zeta)$ , where  $\zeta$  is a primitive root of unity. This allows us to represent an algebraic number as an element in  $\mathbb{Q}[\zeta]$ .

**Remark 0.0.3.** For many number fields, integral bases are known. However, computing an integral basis of a number field is, in general, not an easy problem, as it involves factorization of a large integer [5].

It is well-known that for a Pisot number  $\alpha$ ,  $\alpha^n$  is exponentially close to an integer as  $n$  grows. In this paper, we investigate the problem of computing the integer  $[\alpha^n]$  and its remainder modulo a positive integer  $m$ , where  $[*]$  denotes the function to the nearest integer. Modular exponentiation is the most important operation in implementation of a public key cryptography. By using the repeated squaring algorithm,  $a^n$  can be computed using only  $O(\log n)$  many multiplications, and hence,  $a^n \pmod{m}$  can be computed efficiently if  $a$  is an integer. However, if the base of the exponentiation  $a$  is not an integer, then the problem of computing  $[\alpha^n]$  is considered to be hard. Note that  $[\alpha^n]$  can be too large to be outputted, but in many cases, we are interested in the number of basic operations in integers to produce the number, regardless to the size of the operands. To this end, Tau functions were introduced to measure the complexity of an integer [3].

**Definition 0.0.4.** A straight-line program to compute an integer  $n \in \mathbb{N}$  is a sequence of ring operations (namely, addition, subtraction and multiplication) to produce the integer  $n$  from the constant 1. Let  $\tau(n)$  be the length of the shortest straight-line program computing  $n$ . For a sequence of integers  $x_1, x_2, \dots, x_i, \dots$ , if there exists a polynomial  $p$  such that  $\tau(x_n) \leq p(\log n)$ , then the sequence of integers is called easy to compute. Otherwise, we say that the sequence is hard to compute.

Many well-known integer sequences are conjectured to be hard to compute, e.g.  $n!$ . Pascal Koiran [10] conjectured that the sequences  $\lfloor 2^n \sqrt{2} \rfloor$  and  $\lfloor (3/2)^n \rfloor$  are also hard to compute. Here we show that, on the contrary, a similar sequence  $[\alpha^n]$  is easy to compute if  $\alpha$  is a Pisot number. Namely, we show the following:

**Theorem 0.0.5.** For a fixed Pisot number  $\alpha$ , we can find a straight-line program of length  $O(\log n)$  for  $[\alpha^n]$  in time  $(\log n)^{O(1)}$ . Hence,

$$\tau([\alpha^n]) = O(\log n).$$

As a corollary, we prove that the problem of computing the modular exponentiation of a Pisot number is easy. More precisely,

**Corollary 0.0.6.** Given a Pisot number  $\alpha$ , and two positive integers  $m$  and  $n$ , there exists an algorithm to compute  $[\alpha^n] \pmod{m}$  in time  $(\log(mn))^{O(1)}$ .

The paper proceeds as follows: Section 1 demonstrates the first algorithm to determine a Pisot number generating a given real algebraic field and proves the Theorem 0.0.1. Section 2 describes the algorithms to find a straight-line program for  $[\alpha^n]$  and to compute  $[\alpha^n] \pmod{m}$  of a given Pisot number  $\alpha$  and proves the Theorem 0.0.5 and Corollary 0.0.6.

**Notations:** Let the lowercase letters in bold and the capital letters in bold represent vectors and matrices, respectively.

# 1. AN ALGORITHM TO SEARCH A PISOT NUMBER IN A TOTALLY REAL NUMBER FIELD

**1.1. Preliminaries.** Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space. A (full rank) integral lattice is the set

$$\mathcal{L} = \left\{ \sum_{i=1}^n x_i \mathbf{b}_i \mid x_i \in \mathbb{Z} \right\},$$

where  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  are linearly independent vectors over  $\mathbb{R}$  and  $\mathbf{b}_i \in \mathbb{Z}^n$  for  $1 \leq i \leq n$ . The determinant of the lattice is defined to be the absolute value of the determinant of the matrix  $(b_{ij})$ , where  $b_{ij}$  is the  $j$ -th coordinate of  $\mathbf{b}_i$ .

Minkowski's convex body theorem (page 12 in [11]) asserts that given any convex set in  $\mathbb{R}^n$ , which is symmetric with respect to the origin and with volume greater than  $2^n \det(\mathcal{L})$ , there exists a non-zero lattice point in the set. As a corollary, Minkowski's first theorem says that the length of the shortest vector in  $\mathcal{L}$  satisfies

$$\lambda_1 < \sqrt{n} \det(\mathcal{L})^{1/n}.$$

While no known efficient algorithm can find the shortest vector, or even a vector within the Minkowski's bound, there are polynomial time algorithms to approximate the shortest vector in a lattice. The Lenstra-Lenstra-Lovasz (LLL) algorithm can find in polynomial time a vector whose length is at most  $(2/\sqrt{3})^n$  times the length of the shortest vector of a lattice (see page 33 in [11]). The Block-Korkine-Zolotarev (BKZ) algorithm can achieve a better approximation factor. In this paper, we use the LLL reduction algorithm, which is adequate for our purpose.

**1.2. The problem and the idea.** Let  $\mathbb{F}$  be a real algebraic field and let  $\beta_1, \dots, \beta_k$  be its integral basis. Each algebraic integer in  $\mathbb{F}$  can be represented as

$$\alpha = z_1 \beta_1 + \dots + z_k \beta_k,$$

where  $z_1, z_2, \dots, z_k$  are rational integers, and its conjugates are

$$\sigma_i(\alpha) = z_1 \sigma_i(\beta_1) + \dots + z_k \sigma_i(\beta_k), (1 \leq i \leq k-1),$$

where each  $\sigma_i (1 \leq i \leq k-1)$  is a field automorphism of  $\mathbb{F}$ .

Let us consider the lattice  $\mathcal{L}$  generated by the  $k$  column vectors of the following matrix:

$$\mathbf{D} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \\ \sigma_1(\beta_1) & \sigma_1(\beta_2) & \cdots & \sigma_1(\beta_k) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k-1}(\beta_1) & \sigma_{k-1}(\beta_2) & \cdots & \sigma_{k-1}(\beta_k) \end{bmatrix}.$$

We note that the square of the determinant of the matrix  $\mathbf{D}$  is the discriminant of the field  $\mathbb{F}$ . Each column of  $\mathbf{D}$  consists of one element of the integral basis and its conjugates, thus each vector in the lattice  $\mathcal{L}$  corresponds to an algebraic integer of  $\mathbb{F}$  given by its first element.

It can be proved that there exist Pisot numbers in the field  $\mathbb{F}$  by applying Minkowski's theorem on the lattice  $\mathcal{L}$  [12] and [13, Page 3]. Furthermore, we can derive an upper bound of the minimal Pisot number from the above proof. For the completeness, we include the modified proof below:

**Lemma 1.2.1.** [13] *Let  $\mathbb{F}$  be a real algebraic field with discriminant  $\Delta_{\mathbb{F}}$ . Given a real number  $0 < \delta < 1$ , there exists a Pisot number  $\alpha$  bounded by  $B = \frac{\sqrt{|\Delta_{\mathbb{F}}|}}{\delta^{k-1}}$  such that  $\mathbb{Q}[\alpha] = \mathbb{F}$ .*

*Proof.* Firstly, we show the existence of Pisot numbers in the field. For any positive real number  $B$  and  $\delta < 1$ , all the points  $(r_1, r_2, \dots, r_k) \in \mathbb{R}^k$  satisfying

$$|r_1| < B \quad \text{and} \quad |r_i| < \delta, (1 \leq i \leq k-1)$$

form a hyper-cuboid of volume  $2^k B \delta^{k-1}$ . If  $B \delta^{k-1} \geq \sqrt{|\Delta_{\mathbb{F}}|}$ , which can be satisfied by set  $B = \frac{\sqrt{|\Delta_{\mathbb{F}}|}}{\delta^{k-1}}$ . Then by Minkowski's convex body theorem, there exists a nonzero lattice point of  $\mathcal{L}$  in the convex body. In other words, there exists rational integers  $z_1, \dots, z_k$  such that

$$|z_1 \beta_1 + \dots + z_k \beta_k| \leq B,$$

$$|z_1 \sigma_i(\beta_1) + \dots + z_k \sigma_i(\beta_k)| \leq \delta < 1 \quad (1 \leq i \leq k-1).$$

Hence, the algebraic integer

$$\alpha = z_1 \beta_1 + \dots + z_k \beta_k$$

is a Pisot number by definition.

Next, we need to show that  $\mathbb{Q}[\alpha] = \mathbb{F}$ . By definition,  $\alpha$  is greater than any of its conjugates. If we denote  $e = [\mathbb{Q}[\alpha] : \mathbb{Q}]$ ,  $f = [\mathbb{F} : \mathbb{Q}]$  and suppose  $e < f$ , then we have  $e|f$  and let  $f = ed$  for some  $d \in \mathbb{Z}$ . Thus,  $\alpha$  will appear  $d$  times in its Galois conjugates, which is a contradiction with the definition of the Pisot number.  $\square$

**1.3. The algorithm and its correctness.** The above lemma demonstrates the existence of the Pisot number in a real algebraic field. However, the proof is non-constructive; it does not provide an efficient method to find a Pisot number. The key idea of our algorithm below is to construct a new lattice similar to  $\mathcal{L}$  and to convert the problem of determining a Pisot number in a given total real field into the problem of finding a vector in the lattice, whose length approximates the shortest vector.

Let  $P$  be a positive real number, we construct another lattice  $\mathcal{L}_P$  generated by the  $k$  column vectors of the following matrix:

$$(1.3.1) \quad \mathbf{D}_P = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_k \\ P\sigma_1(\beta_1) & P\sigma_1(\beta_2) & \dots & P\sigma_1(\beta_k) \\ \vdots & \vdots & \ddots & \vdots \\ P\sigma_{k-1}(\beta_1) & P\sigma_{k-1}(\beta_2) & \dots & P\sigma_{k-1}(\beta_k) \end{bmatrix}.$$

Note that

$$\mathbf{D} = \mathbf{D}_1 \quad \text{and} \quad \det(\mathbf{D}_P) = P^{k-1} \det(\mathbf{D}).$$

We observe:

- From Minkowski's Theorem, we conclude that there is vector in  $\mathcal{L}_P$  with length at most  $\sqrt{k} \sqrt{P^{k-1} \det(\mathbf{D})}$ ;
- On the other hand, if a vector is not corresponding to a Pisot number, then its length is at least  $P$ .

So we can choose an appropriate  $P$  such that the gap between  $\sqrt[k]{P^{k-1} \det(\mathbf{D})}$  and  $P$  is large enough, then the LLL algorithm can find a short vector of length less than  $P$ , which must correspond to a Pisot number. Our algorithm can be described as follows:

**Algorithm 1**

Input: Integral basis  $\beta_1, \dots, \beta_k$  of a real Galois extension  $\mathbb{F}$  over  $\mathbb{Q}$ .

- (1) Compute  $P$  to be an integer bigger than  $(\frac{2}{\sqrt{3}})^{k^2} k^{k/2} \det(\mathbf{D})$ ;
- (2) Construct the basis of the lattice  $\mathcal{L}_P$  as the columns of the matrix  $\mathbf{D}_P$  defined by Equation (1.3.1);
- (3) Run LLL algorithm on the basis of  $\mathcal{L}_P$ ;
- (4) Recover the Pisot number which is the absolute value of the first element of the returned approximate shortest vector.

Output: A Pisot number.

Now we proceed to prove Theorem 0.0.1. We need to show that the proposed algorithm is correct and it runs in polynomial time of the input size.

*Proof.* (of Theorem 0.0.1) Firstly, we need to show that this algorithm returns a Pisot number. According to Minkowski's Theorem, there is a nonzero vector in  $\mathcal{L}_P$  of length at most  $\sqrt[k]{P^{k-1} \det(\mathbf{D})}$ . On the other hand, for any algebraic integer  $x_1\beta_1 + \dots + x_k\beta_k$  in  $\mathbb{F}$  that is not a Pisot number, the vector

$$x_1 \begin{pmatrix} \beta_1 \\ P\sigma_1(\beta_1) \\ \vdots \\ P\sigma_{k-1}(\beta_1) \end{pmatrix} + x_2 \begin{pmatrix} \beta_2 \\ P\sigma_1(\beta_2) \\ \vdots \\ P\sigma_{k-1}(\beta_2) \end{pmatrix} + \dots + x_k \begin{pmatrix} \beta_k \\ P\sigma_1(\beta_k) \\ \vdots \\ P\sigma_{k-1}(\beta_k) \end{pmatrix}$$

in the lattice has length at least  $P$ , since there exists  $1 \leq i \leq k-1$  such that

$$|x_1 P\sigma_i(\beta_1) + \dots + x_k P\sigma_i(\beta_k)| = P|x_1\sigma_i(\beta_1) + \dots + x_k\sigma_i(\beta_k)| \geq P.$$

If we set  $P > (\frac{2}{\sqrt{3}})^{k^2} k^{k/2} \det(\mathbf{D})$ , then

$$P/(\sqrt[k]{P^{k-1} \det(\mathbf{D})}) > (\frac{2}{\sqrt{3}})^k,$$

then the vector returned by the LLL algorithm will have length less than  $P$ , which must correspond to a Pisot number.

Next, we need to show that the returned Pisot number  $\alpha$  is a primitive element of the field. By definition,  $\alpha$  is greater than any of its conjugates. If we denote  $e = [\mathbb{Q}[\alpha] : \mathbb{Q}]$ ,  $f = [F : \mathbb{Q}]$  and suppose  $e < f$ , then we have  $e|f$  and let  $f = ed$  for some  $d \in \mathbb{Z}$ . Thus  $\alpha$  will appear  $d$  times in its conjugates, which is a contradiction with the definition of the Pisot number.

At last, we analyze the running time of the algorithm. First we observe that  $\log P = (k \det(\mathbf{D}))^{O(1)}$ . The most costly part of the algorithm is Step 3, where the LLL algorithm runs in polynomial time of the size of the lattice basis. Thus the overall time of the algorithm is polynomial in the size of the integral basis.  $\square$

**Remark 1.3.1.** Given a real number  $\varepsilon > 0$ , if we choose  $P$  such that

$$P > (\frac{2}{\sqrt{3}})^{k^2} k^{k/2} \det(\mathbf{D})/\varepsilon^k,$$

then we have

$$\varepsilon P > \left(\frac{2}{\sqrt{3}}\right)^{k^2} k^{k/2} \det(\mathbf{D}).$$

Hence, the algorithm actually determines an  $\varepsilon$ -Pisot number in this case.

**1.4. Examples.** In the following examples, we use the lattice functions in Victor Shoup's NTL package.

**Example 1.4.1.** Let us illustrate our algorithm by taking the field  $\mathbb{Q}(2 \cos \frac{2\pi}{15})$  as an example. The extension degree  $k = [\mathbb{Q}(2 \cos \frac{2\pi}{15}) : \mathbb{Q}] = \frac{\phi(15)}{2} = 4$  and an integral basis is given by

$$\beta_1 = 2 \cos \frac{2\pi}{15}, \beta_2 = 2 \cos \frac{4\pi}{15}, \beta_3 = 2 \cos \frac{8\pi}{15}, \beta_4 = 2 \cos \frac{14\pi}{15}.$$

- (1) Choose  $\varepsilon = 0.5$ , compute  $P = 85769 > (\frac{2}{\sqrt{3}})^{16} * 4^2 * \sqrt{1125} * 16$ ;
- (2) Construct the basis of  $\mathcal{L}_P$  as the column vectors of  $\mathbf{D}_P$ ;
- (3) Run LLL algorithm over the basis of  $\mathcal{L}_P$ ;
- (4) Recover the Pisot number which is the following:

$$\alpha = 2105\beta_1 + 1215\beta_2 + 1440\beta_3 + 139\beta_4.$$

**Remark 1.4.2.** We note that in the second step we first compute  $P\sigma_i(\beta_j)$  then take the integer part as the input of the matrix. And we can check that the Galois conjugates of the returned number are:  $-0.063765\dots$ ,  $0.065726\dots$ , and  $-0.048703\dots$

**Example 1.4.3.** Now let's look at another example, the field  $\mathbb{Q}(2 \cos \frac{2\pi}{17})$ . The extension degree  $k = [\mathbb{Q}(2 \cos \frac{2\pi}{17}) : \mathbb{Q}] = \frac{\phi(17)}{2} = 8$  and one integral basis is given by

$$\begin{aligned} \beta_1 &= 2 \cos \frac{2\pi}{17}, \beta_2 = 2 \cos \frac{4\pi}{17}, \beta_3 = 2 \cos \frac{6\pi}{17}, \beta_4 = 2 \cos \frac{8\pi}{17}, \\ \beta_5 &= 2 \cos \frac{10\pi}{17}, \beta_6 = 2 \cos \frac{12\pi}{17}, \beta_7 = 2 \cos \frac{14\pi}{17}, \beta_8 = 2 \cos \frac{16\pi}{17}. \end{aligned}$$

- (1) Compute  $P = 825982306366 > (\frac{2}{\sqrt{3}})^{64} * 8^4 * \sqrt{410338673}$ ;
- (2) Construct the basis of  $\mathcal{L}_P$  as the column vectors of  $\mathbf{D}_P$ ;
- (3) Run LLL algorithm over the basis of  $\mathcal{L}_P$ ;
- (4) Recover the Pisot number which is the following:

$$\begin{aligned} \alpha &= -24708871\beta_1 - 95498414\beta_2 - 202808109\beta_3 - 332145187\beta_4 \\ &\quad - 466041959\beta_5 - 586414924\beta_6 - 677007046\beta_7 - 725583357\beta_8. \end{aligned}$$

**Remark 1.4.4.** We can compute the Galois conjugates of the returned number are:  $0.039500\dots$ ,  $0.048267\dots$ ,  $0.064900\dots$ ,  $-0.019990\dots$ ,  $-0.057987\dots$ ,  $0.062209\dots$  and  $0.036031\dots$

## 2. AN ALGORITHM TO COMPUTE MODULAR EXPONENTIAL OF A PISOT NUMBER

**2.1. The problem and the idea.** Given a Pisot number  $\alpha$  of degree  $d$  and its minimal polynomial over  $\mathbb{Q}$

$$f(x) = x^d + c_{d-1}x^{d-1} + \dots + c_1x + c_0,$$

we want to determine a straight-line program for  $[\alpha^n]$  and then to compute  $[\alpha^n] \bmod m$ , where  $n, m$  are given positive numbers.

**Lemma 2.1.1.** *Given a Pisot number  $\alpha_1$  of degree  $d$  with conjugates  $\alpha_2, \dots, \alpha_d$ ,  $|\alpha_2| \geq |\alpha_i|$ ,  $3 \leq i \leq d$ . If  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , then*

$$[\alpha_1^n] = [\alpha_1^n + \alpha_2^n + \dots + \alpha_d^n] = \alpha_1^n + \alpha_2^n + \dots + \alpha_d^n.$$

*Proof.* Suppose  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , we have

$$|\alpha_2^n + \dots + \alpha_d^n| \leq (d-1)|\alpha_2|^n < \frac{1}{2}.$$

Note that for any given positive integer  $n$ ,  $\alpha_1^n + \alpha_2^n + \dots + \alpha_d^n$  is an integer itself. Thus we deduce if  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , then

$$[\alpha_1^n] = [\alpha_1^n + \alpha_2^n + \dots + \alpha_d^n] = \alpha_1^n + \alpha_2^n + \dots + \alpha_d^n.$$

□

Lemma 2.1.1 shows that we can convert the problem of finding  $[\alpha^n]$  of a Pisot number  $\alpha$  to the computation of  $\alpha^n + \alpha_2^n + \dots + \alpha_d^n$  when  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , where  $\alpha_2, \alpha_3, \dots, \alpha_d$  are conjugates of  $\alpha$ . For the sake of consistency, we will sometimes write  $\alpha_1$  in the place of  $\alpha$  below.

**2.2. Notations and preliminaries.** Let the polynomial  $f(x) = x^d + c_{d-1}x^{d-1} + \dots + c_1x + c_0$  be the minimal polynomial for a Pisot number  $\alpha$  over  $\mathbb{Q}$ . The companion matrix [8] of the polynomial  $f(x)$  is defined by

$$\mathbf{C}(f) = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{d-1} \end{bmatrix}.$$

Since  $f(x)$  is irreducible over  $\mathbb{Q}[x]$ , it has distinct roots  $\alpha_1, \alpha_2, \dots, \alpha_d$ . Thus the companion matrix is diagonalizable as follows:

$$\mathbf{V}\mathbf{C}(f)\mathbf{V}^{-1} = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{bmatrix},$$

where all the non-diagonal elements are zero and  $\mathbf{V}$  represents the Vandermonde matrix corresponding to the  $\alpha_i$ :

$$\mathbf{V} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{d-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_d & \alpha_d^2 & \dots & \alpha_d^{d-1} \end{bmatrix}.$$

**2.3. The algorithm and its correctness.** Given a Pisot number  $\alpha$  of degree  $d$  with conjugates  $\alpha_2, \dots, \alpha_d$  and its minimal polynomial over  $\mathbb{Q}$

$$f(x) = x^d + c_{d-1}x^{d-1} + \dots + c_1x + c_0,$$

firstly, we determine  $\tau([\alpha^n])$ , where  $n$  are given positive numbers.

**Algorithm 2**

Input: A Pisot number  $\alpha$  with conjugates  $\alpha_2, \dots, \alpha_d$  and its minimal polynomial over  $\mathbb{Q}$ :  $f(x) = x^d + c_{d-1}x^{d-1} + \dots + c_1x + c_0$  and a positive integer  $n$ .

- (1) If  $n \leq \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , compute  $[\alpha_1^n]$  directly;
- (2) If  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ ,
  - (a) Construct  $\mathbf{C}(f)$ ;
  - (b) Find a straight-line program for every entry of  $\mathbf{C}^n(f)$  utilizing the repeated squaring algorithm;
  - (c) Compute the trace of  $\mathbf{C}^n(f)$ .

Output: A straight-line program of computing  $[\alpha^n]$ .

Now we proceed to prove Theorem 0.0.5, namely, we need to show that the proposed algorithm is correct, and the number of basic operations involved is polynomial in the input size.

*Proof.* ( of Theorem 0.0.5 ) Firstly, we show that the algorithm is correct. When  $n > \log_{|\alpha_2|} \frac{1}{2(d-1)}$ , by Lemma 3.1, we have

$$[\alpha^n] = \alpha_1^n + \dots + \alpha_d^n.$$

Since the conjugates of  $\alpha$  are distinct, the companion matrix of  $f(x)$  can be diagonalized as

$$\mathbf{V}\mathbf{C}(f)\mathbf{V}^{-1} = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{bmatrix},$$

where all the non-diagonal elements are zero and  $\mathbf{V}$  represents the Vandermonde matrix corresponding to the  $\alpha_i$ . We have

$$(\mathbf{V}\mathbf{C}(f)\mathbf{V}^{-1})^n = \begin{bmatrix} \alpha_1^n & & \\ & \ddots & \\ & & \alpha_d^n \end{bmatrix}.$$

Because

$$(\mathbf{V}\mathbf{C}(f)\mathbf{V}^{-1})^n = \mathbf{V}\mathbf{C}^n(f)\mathbf{V}^{-1},$$

we have

$$\begin{aligned} \text{tr}(\mathbf{V}\mathbf{C}^n(f)\mathbf{V}^{-1}) &= \text{tr}((\mathbf{V}\mathbf{C}(f)\mathbf{V}^{-1})^n) \\ &= \alpha_1^n + \dots + \alpha_d^n \\ &= [\alpha^n], \end{aligned}$$

where  $\text{tr}$  is the trace function of the matrix. Furthermore, we have

$$\text{tr}(\mathbf{C}^n(f)) = \text{tr}(\mathbf{V}\mathbf{C}^n(f)\mathbf{V}^{-1}),$$

hence

$$\text{tr}(\mathbf{C}^n(f)) = [\alpha^n].$$

Next, we analyze the number of basis operations needed. Since the computation of the the matrix  $\mathbf{C}^n(f)$  takes  $O(\log n)$  matrix multiplications and other steps take constant number of operations, we have

$$\tau([\alpha^n]) = O(\log n).$$

□



We can modify the last algorithm to compute the modular exponentiation of a Pisot number as follows:

**Algorithm 3**

Input: A Pisot number  $\alpha$  of degree  $d$  given by its minimal polynomial over  $\mathbb{Q}$ :  $f(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1x + c_0$ , two positive integers  $m, n$ .

- (1) Construct a straight-line program of length  $O(\log n)$  for  $[\alpha^n]$ ;
- (2) Evaluate the straight-line program in the ring  $\mathbb{Z}/m\mathbb{Z}$ .
- (3) Output the last step of the straight-line program.

Output:  $[\alpha^n] \bmod m$ .

Sketch of the proof of Corollary 0.0.6: we need to show that the proposed algorithm is correct and it runs in polynomial time of the input size. The proof is similar with the proof of Theorem 0.0.5 except that here we compute  $\mathbf{C}^n(f) \bmod m$  instead of  $\mathbf{C}^n(f)$  which makes it run in time  $(\log(mn))^{O(1)}$ .

### 3. CONCLUDING REMARKS

In this paper, we present two deterministic polynomial time algorithms about certain computations of Pisot numbers. The first one is to search a Pisot number  $\alpha$  such that  $\mathbb{Q}[\alpha] = \mathbb{F}$  given a real Galois extension  $\mathbb{F}$  of  $\mathbb{Q}$  with integral basis. We remark that we can find Pisot numbers with high degree utilizing the algorithm. The second one is to compute the modular exponentiation of a Pisot number.

### REFERENCES

- [1] A.K.Lenstra, J.H.W.Lenstra, L. Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261(4):515–534, 1982.
- [2] M.J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.P. Schreiber. *Pisot and Salem Numbers*. Birkhäuser Verlag, 1992.
- [3] Lenore Blum, Mike Shub, Steve Smale. On a theory of computation and complexity over the real numbers: *NP*-completeness, recursive functions and universal machines. *Bulletin of the American Mathematical Society*, 21(1), 1989.
- [4] David W. Boyd. Pisot and salem numbers in intervals of the real line. *Mathematics of Computation*, 32:1244–1260, 1978.
- [5] H. Cohen. *A Course In Computational Algebraic Number Theory*, *Graduate Texts in Mathematics*. Springer-Verlag, 1996.
- [6] J. Dufresnoy, Ch. Pisot. Étude de certaines fonctions méromorphes bornées sur le cercle unité, application à un ensemble fermé d’entiers algébriques. *Ann. Sci. École Norm. Sup.*, 72:69–92, 1955.
- [7] Ai-Hua Fan, Jörg Schmeling. Pisot numbers in any real algebraic number field are relatively dense. *Journal of Algebra*, 272:470–475, 2004.
- [8] Roger A. Horn, Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
- [9] J.P.Bell, K.G.Hare. A classification of (some)Pisot-Cyclotomic numbers. *Journal of Number Theory*, 115:215–229, 2005.
- [10] Pascal Koiran. Valliant’s Model and the Cost of Computing Integers. *Computational complexity*, 13:131–146, 2004.
- [11] Daniele Micciancio, Shafi Goldwasser. *Complexity of Lattice Problems: A Cryptographic Perspective*. Kluwer Academic Publishers, 2002.
- [12] C. Pisot. La répartition modulo 1 et les nombres algébriques. *Annali di Pisa*, strony 205–248, 1938.
- [13] Raphael Salem. *Algebraic numbers and Fourier analysis*. D.C.Heath and Company, 1963.

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